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# On Subleading Contributions to the AdS/CFT Trace Anomaly

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Abstract: In the context of the AdS/CFT correspondence, we perform a direct computation in $AdS_5$ supergravity of the trace anomaly of a $d = 4$, $\mathcal{N} = 2$ SCFT. We find agreement with the field theory result up to next to leading order in the $1/N$ expansion. In particular, the order $N$ gravitational contribution to the anomaly is obtained from a Riemann tensor squared term in the 7-brane effective action deduced from heterotic - type I duality. We also discuss, in the AdS/CFT context, the order $N$ corrections to the trace anomaly in $d = 4$, $\mathcal{N} = 4$ SCFTs involving SO or Sp gauge groups.

Keywords: 1/N Expansion, String Duality, Anomalies in Field and String Theories, Supergravity Models.

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## 1. Introduction

Recently, a lot of work has been done on the conjectured [1] AdS/CFT correspondence between string theory or M-theory compactifications on $AdS_{d+1}$ and $d$-dimensional conformal field theories. In particular, this conjecture relates [2, 3] correlation functions of local operators in the conformal field theory to amplitudes in the 'bulk' string theory or M-theory, with the boundary values of the bulk fields interpreted as sources coupling to the operators of the 'boundary' conformal field theory.

An example of particular interest is the conjectured equivalence between $\mathcal{N}=4, d=4$ supersymmetric SU($N$) Yang-Mills theory and type IIB superstring theory

on $AdS_5 \times S^5$ (with $N$ units of RR five-form $F^{(5)}$ flux on $S^5$). This duality identifies the complex gauge coupling constant

$$\tau_{YM} = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \tag{1.1}$$

of the SYM theory with the constant expectation value of the type IIB string coupling

$$\tau_s = \left\langle C^{(0)} + i\mathrm{e}^{-\phi} \right\rangle \equiv \frac{\chi}{2\pi} + \frac{i}{g_s} \,. \tag{1.2}$$

The radius of $S^5$ (or the curvature radius of $AdS_5$) is

$$L = (4\pi g_s N)^{1/4} \ell_s \,, \tag{1.3}$$

with $\ell_s$ the string length, $\ell_s^2 = \alpha'$. In terms of the 't Hooft coupling $\lambda = g_{YM}^2 N$, the dimensionless scale $L^2/\alpha'$ of string theory on $AdS_5 \times S^5$ is related to the SYM parameters by

$$\lambda^{1/2} = \frac{L^2}{\alpha'} \,. \tag{1.4}$$

Correlation functions of string theory on $AdS_5 \times S^5$ are given by a double expansion in $g_s$ and $\alpha'/L^2$, which can be written as a double expansion in terms of $1/N = 4\pi g_s (\alpha'/L^2)^2$ and $\lambda^{-1/2}$. For closed oriented strings this is actually an expansion in even powers of $N$, the string theory tree-level (supergravity) contribution being of order $N^2$.

Correlation functions of the SYM theory, on the other hand, have a $1/N$ expansion, valid when $N$ is large, $g_{YM}^2$ is small, and $\lambda$ is kept finite (and small). For SU($N$) theories with adjoint fields only, this is once again an expansion in even powers of $N$, the leading contributions, of order $N^2$, coming from planar diagrams.

According to the (strong form of the) AdS/CFT correspondence, these two theories should give rise to the same function of $\lambda$ at each order in $N$. However, as one has an expansion in terms of $\lambda$ (weak 't Hooft coupling) and the other in terms of $\lambda^{-1/2}$ (the $\alpha'$ or, better, $\alpha'/L^2$, expansion of string theory), in practice this comparison is restricted to the rather limited set of quantities which are $\lambda$-independent, such as global anomalies.

Leading order $N^2$ contributions to the chiral anomalies were checked in e.g. [3, 4], and trace anomalies were discussed (at the linearized level) in [5] by comparing the bulk supergravity action with the effective action arising from the coupling of $\mathcal{N} = 4$ SYM to $\mathcal{N} = 4$ conformal supergravity.

The complete leading order 'holographic Weyl anomaly' was determined in general in [6]. In particular, it was found there that the leading supergravity contribution to the trace anomaly involves only the squares of the Ricci tensor and Ricci scalar of the boundary metric and not the square of the Riemann tensor itself. This implies



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that conformal field theories with a standard (product space) supergravity dual necessarily have $a = c$ to leading order in $N$, where $a$ and $c$ are the coefficients of the Euler and Weyl terms in the standard expression

$$\langle T^{\mu}_{\mu} \rangle = -aE_4 - cI_4 \tag{1.5}$$

for the conformal anomaly. In [7] the calculations of [6] have been generalized to dilatonic gravity.

The check of subleading corrections in the $1/N^2$ expansion is hampered by the fact that, for closed string theory, these correspond to string loop corrections with RR background fields which are still not well understood.[1]

However, as pointed out in [9], certain subleading $1/N$ corrections (i.e. terms of order $N$) in theories with open or unoriented strings, corresponding to $SO(N)$ or Sp($N$) gauge theories, may be accessible. In [9] an $\mathcal{N} = 2$ superconformal field theory arising from D3-branes on a $\mathbb{Z}_2$ orientifold O7-plane with D7-branes [10] was analyzed. In particular, in a rather subtle analysis it was shown that the order $N$ contribution to the chiral U(1) R-current anomaly, proportional to $(a-c)$, is correctly reproduced in the dual supergravity theory on $AdS_5 \times X^5$, where $X^5 = S^5/\mathbb{Z}_2$ [11], by bulk Chern-Simons couplings on the D7 and O7 world-volumes.

By supersymmetry, this chiral anomaly is related to the trace anomaly and therefore indirectly [9] also confirms the AdS/CFT correspondence for the conformal anomaly to this order in the large $N$ expansion. The purpose of this note is to perform a direct calculation of the trace anomaly along the lines of [6]. By chasing the Chern-Simons couplings from type I' theory back to ten dimensions, we see that they originate from the Green-Schwarz couplings $H^2 = (dB + \omega_L + \cdots)^2$ in the heterotic string. These terms are known to be related by supersymmetry to CP even $R^2$-terms proportional to the Riemann tensor squared and $F^2$-terms for the gauge group SO(8) $\subset$ SO(32) [12, 13].

By using heterotic - type I duality and T-dualizing to type I', we show that these terms give rise to order $N$ Riemann tensor and gauge field strength squared terms in eight dimensions leading to a subleading order $N$ contribution to the conformal anomaly upon reduction to $AdS_5$. We then show that the external gauge field contribution and the crucial coefficient of the Riem$^2$-term of the boundary metric in the conformal anomaly, proportional to $(a - c)$, are precisely reproduced by the supergravity calculation.

We also find other terms of order $N$, proportional to the squares of the Ricci tensor and Ricci scalar. This particular linear combination differs from that of the field theory result precisely by a term of order $N$ attributable to an effective five-dimensional cosmological constant. We have been unable to determine this contribution because of our ignorance regarding other four-derivative terms in the type I'

[1] For a preliminary discussion of some subleading $\mathcal{O}(1)$ contributions to anomalies see [8].



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theory like $(F^{(5)})^4$ and $R(F^{(5)})^2$. Conversely, comparing the supergravity calculation with the known field theory result gives a concrete (but not in itself particularly interesting) prediction for the $1/N$ contribution to the effective cosmological constant in this theory.

Another class of theories with subleading order $N$ corrections to the trace anomaly are $\mathcal{N} = 4$ SYM theories with orthogonal or symplectic gauge groups. These can be realized as low-energy theories on D3-branes at an orientifold O3-plane and a candidate for their supergravity dual is tpye IIB string theory on an $AdS_5 \times \mathbb{RP}^5$ orientifold. At first, these theories appear to present a puzzle as there are no D-branes or O-planes wrapping the $AdS_5$ and therefore there can be no orientifold or open string corrections to the bulk theory. We will show that both the leading and the subleading order $N$ contributions to the anomaly are correctly reproduced by the classical Einstein action by taking into account the (fractional) RR charge of the O3-plane.

In section 2, we review the CFT side of the gravitational and external gauge field contributions to the conformal anomaly. In section 3, we deduce the relevant $R^2$ and $F^2$ terms in the $AdS_5$ supergravity action via heterotic - type I - type I' duality. In section 4, we review the calculation of the leading $\mathcal{O}(N^2)$ contribution to the trace anomaly following [6]. We then deduce the $\mathcal{O}(N)$ contributions by extending the analysis of [6] to include the $R^2$ and $F^2$ terms (section 5). In section 6 we discuss the $\mathcal{N} = 4$ SYM theories for SO and Sp gauge groups, and we conclude with a discussion of the 'missing' contributions due to an effective cosmological constant of order $N$.

## 2. The trace anomaly on the CFT side

The field theory of interest is [9] an $\mathcal{N} = 2$ superconformal field theory with $\mathrm{Sp}(N)$ gauge group, and 4 fundamental and one antisymmetric traceless hypermultiplet. It arises [10] as the low-energy theory on the world volume on $N$ D3-branes sitting inside eight D7-branes at an O7-plane. Among the global symmetries of the theory there are an SO(8)-symmetry (from the D7-branes) as well as an $\mathrm{SU}(2) \times \mathrm{U}(1)$ R-symmetry of the $\mathcal{N} = 2$ superconformal algebra. Taking the near-horizon limit of this configuration one finds [11] that the conjectural string theory dual of this theory is type IIB string theory on $AdS_5 \times X^5$ where $X^5 = S^5/\mathbb{Z}_2$ in which the D7 and O7 fill the $AdS_5$ and are wrapped around an $S^3$ which is precisely the fixed point locus of the $\mathbb{Z}_2$. Because of the $\mathbb{Z}_2$ action, the relation between the five-form flux $N$ and the curvature radius of $AdS_5$ is now

$$L = (8\pi g_s N)^{1/4} \ell_s \tag{2.1}$$

instead of (1.3). We will set $\ell_s = 1$ in the following.

The trace anomaly, when the theory is coupled to an external metric, is

$$\langle T^{\mu}_{\mu} \rangle = -aE_4 - cI_4 \,, \tag{2.2}$$



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where, using shorthand notation,

$$\mathrm{Riem}^2 = R_{ijkl}R^{ijkl}\,, \tag{2.3}$$

with $R_{ijkl}$ the Riemann curvature tensor of the metric of the (boundary) space-time etc.,

$$\begin{aligned} E_4 &= \frac{1}{16\pi^2}\left(\mathrm{Riem}^2 - 4\mathrm{Ric}^2 + \mathrm{R}^2\right) \\ I_4 &= -\frac{1}{16\pi^2}\left(\mathrm{Riem}^2 - 2\mathrm{Ric}^2 + \frac{1}{3}\mathrm{R}^2\right) \end{aligned} \tag{2.4}$$

Thus

$$\langle T^\mu_{\ \mu}\rangle = \frac{1}{16\pi^2}\left[(c-a)\mathrm{Riem}^2 + (4a-2c)\mathrm{Ric}^2 + \left(\frac{1}{3}c - a\right)\mathrm{R}^2\right]. \tag{2.5}$$

and we see that the $\mathrm{Riem}^2$-term is proportional to $(c-a)$.

The coefficients $a$ and $c$ are determined in terms of the field content of the theory. In particular, for the vector- and hypermultiplets of the $\mathcal{N}=2$ theories one has

$$\begin{aligned} a_V &= \frac{5}{24}\,, \qquad & c_V &= \frac{1}{6}\,, \\ a_H &= \frac{1}{24}\,, \qquad & c_H &= \frac{1}{12}\,. \end{aligned} \tag{2.6}$$

Thus, for one $\mathcal{N}=4$ multiplet ($n_V = n_H = 1$) one has $a = c = 1/4$ and the trace anomaly of $\mathcal{N}=4$ SU($N$) SYM theory is

$$\langle T^\mu_{\ \mu}\rangle = \frac{N^2-1}{32\pi^2}\left[\mathrm{Ric}^2 - \frac{1}{3}\mathrm{R}^2\right]. \tag{2.7}$$

For an $\mathcal{N}=2$ theory with $n_V$ vector multiplets and $n_H$ hypermultiplets one has

$$\langle T^\mu_{\ \mu}\rangle = \frac{1}{24\times 16\pi^2}\left[(n_H - n_V)\mathrm{Riem}^2 + 12n_V\mathrm{Ric}^2 - \frac{1}{3}(11n_V + n_H)\mathrm{R}^2\right]. \tag{2.8}$$

In the present case, with

$$\begin{aligned} n_V &= N(2N+1) = 2N^2 + N \\ n_H &= 4\times 2N + N(2N-1) - 1 = 2N^2 + 7N - 1 \end{aligned} \tag{2.9}$$

one finds

$$\begin{aligned} a_{total} &\equiv n_V a_V + n_H a_H = \frac{1}{24}(12N^2 + 12N - 1) \\ c_{total} &\equiv n_V c_V + n_H c_H = \frac{1}{24}(12N^2 + 18N - 2) \end{aligned} \tag{2.10}$$

and

$$a_{total} - c_{total} = \frac{1}{24}(1-6N)\,, \tag{2.11}$$



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and the conformal anomaly is

$$\langle T^{\mu}_{\mu} \rangle = \frac{1}{24 \times 16\pi^2} \left[ (6N-1)\mathrm{Riem}^2 + (24N^2+12N)\mathrm{Ric}^2 - \left(8N^2+6N-\frac{1}{3}\right)\mathrm{R}^2 \right]. \tag{2.12}$$

The leading $\mathcal{O}(N^2)$ contribution is

$$\frac{N^2}{16\pi^2}\left[\mathrm{Ric}^2 - \frac{1}{3}\mathrm{R}^2\right]. \tag{2.13}$$

This is exactly twice the $\mathcal{N}=4$ result and this is in accordance with the expected [6, 14] relation between volumes, $\mathrm{Vol}(S^5)$ versus $\mathrm{Vol}(X^5)$, and the leading contribution to the anomaly. On the supergravity side this term arises [6] from a regularization of the (divergent) classical gravity action which is just a volume integral in this case as the $AdS$ scalar curvature (the Einstein-Hilbert lagrangean) is constant for $AdS_5$. We will review this calculation below.

The subleading $\mathcal{O}(N)$-term is

$$\frac{6N}{24 \times 16\pi^2}[\mathrm{Riem}^2 + 2\mathrm{Ric}^2 - \mathrm{R}^2]. \tag{2.14}$$

We will show that, modulo undetermined volume terms of the form (2.13) (with coefficients of order $N$ rather than $N^2$), this term arises from a Riemann curvature squared term in the bulk gravity action (with the precise numerical coefficient deduced from that appearing in the heterotic string through heterotic - type I - type I' duality).

One can also couple the theory to external gauge fields of a flavour symmetry group $G$. In general, the contribution of gauge fields to the trace anomaly has been shown in [15] to be proportional to the beta-function of the corresponding gauge coupling constant. The result obtained in [15] is

$$\langle T^{\mu}_{\mu} \rangle_G = \frac{\beta(g)}{2g} F^a_{ij} F^{a\,ij}, \tag{2.15}$$

where $\beta(g)$ has the standard form

$$\frac{\beta(g)}{2g} = -\frac{g^2}{32\pi^2}\left[\frac{11}{3}c_2(G) - \frac{4}{3}T(R_f) - \frac{1}{3}T(R_s)\right] + \mathcal{O}(g^4). \tag{2.16}$$

Here $R_{f,s}$ are the representations of $G$ on the (Dirac) fermions and (complex) scalars respectively, and $T(R)$ is the Dynkin index of the representation $R$,

$$\mathrm{Tr}_R\, t_a t_b = T(R)\delta_{ab}. \tag{2.17}$$

To apply this result in the present situation we note the following. First of all, in Euclidean space there is a minus sign on the right hand side of (2.15), as can be



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seen by tracing through the derivation in [15, section 3]. Moreover, for an external gauge field, the first term on the right hand side of (2.16) is of course absent. In the present case, we can choose $G = \mathrm{SO}(8)$, and the only fields that are charged under $G$ are the $8N$ fundamental hypermultiplets in the fundamental representation of SO(8). As an $\mathcal{N} = 2$ hypermultiplet in four dimensions consists of one Dirac fermion and two complex scalars, the contribution of external SO(8) gauge fields to the trace anomaly is

$$\langle T^{\mu}_{\mu} \rangle_G = -\frac{NT(\mathbf{8})}{16\pi^2} F^a_{ij} F^{a\,ij} \,. \tag{2.18}$$

We have dropped the factor $g^2$ because we will be working with the scaled gauge fields in terms of which the action takes the form $S = (1/4g^2) \int F^2 + \cdots$.

We see that this term is also of order $N$, and we will show that this contribution to the anomaly is reproduced precisely by an $F^2$-term in the heterotic - type I action or, alternatively, by the $F^2$-term of the Dirac-Born-Infeld D7-brane action (wrapped on the $S^3 \subset X^5$).

## 3. The $R^2$ and $F^2$ terms

In [9], the relevant Chern-Simons terms in the $AdS_5$ bulk action arose from terms proportional to

$$\int C^{(4)} \wedge \mathrm{Tr}(\Omega \wedge \Omega) \,, \int C^{(4)} \wedge \mathrm{Tr}(F \wedge F) \,, \tag{3.1}$$

in the world volume theory of the D7-branes and O7-planes, where $\Omega$ is the Riemann curvature two-form, $F$ denotes the SO(8) gauge field and $C^{(4)}$ is the RR 4-form coupling to the D3-brane. T-dualizing these terms to type I, the RR 4-form becomes a six-form $C^{(6)}$ coupling to the type I 5-brane. Writing this interaction as $F^{(7)} \wedge \omega_{L,YM}$, where $\omega_{L,YM}$ is the Lorentz/Yang-Mills Chern-Simons term, and dualizing $F^{(7)} = *dB$, we see that this term arises from the modification

$$H^2 = (dB + \lambda_L \omega_L - \lambda_{YM} \omega_{YM})^2 \tag{3.2}$$

of the $B$-field kinetic term in the type I and heterotic supergravity actions required by the Green-Schwarz anomaly cancellation mechanism.

Now it is known [12, 13] that supersymmetry relates this term to a four-derivative CP even term $R_{LMNP} R^{LMNP}$ in the ten-dimensional heterotic action together with the SO(32) Yang-Mills term. The relevant part of the heterotic action for our purposes is thus

$$S_h = \frac{1}{16\pi(8\pi^6)} \int d^{10}x \sqrt{G^h} \mathrm{e}^{-2\phi^h} \left( R + \frac{1}{4} \left( R_{LMNP} R^{LMNP} - F^a_{MN} F^{a\,MN} \right) \right) . \tag{3.3}$$

We will now first check explicitly that these two terms in the end give rise to terms of order $N^2$ and $N$ in the AdS supergravity action respectively. We will determine the precise numerical factors below.



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First of all, using the rules of heterotic - type I duality,

$$\begin{aligned} \phi^h &= -\phi^I \\ G^h_{MN} &= \mathrm{e}^{-\phi^I} G^I_{MN} \,, \end{aligned} \tag{3.4}$$

in the type I theory one obtains

$$\int d^{10}x\sqrt{G^I}\left(\mathrm{e}^{-2\phi^I}R + \frac{1}{4}\mathrm{e}^{-\phi^I}\left(R_{LMNP}R^{LMNP} - F^a_{MN}F^{a\,MN}\right)\right). \tag{3.5}$$

Notice that the Riemann tensor square term comes with $\exp(-\phi)$ rather than with $\exp(-2\phi)$. This indicates that it arises from disc (D9-branes) and crosscap (orientifold O9-planes) world-sheets and not from the sphere. The latter was to be expected since the sphere calculation is identical to that in IIB where one knows that there is no $R^2$-term. The observation that type I - heterotic duality dictates the appearance of an $R^2$-contribution from the disc diagram in type I theory was originally made in [16].

For a constant dilaton, which is all that we are interested in, the dependence of the action on the type I string coupling constant $g_I$ (defined in general by $g_s = \exp < \phi_s >$, for $s = h, I, I'$ respectively) is thus

$$S_I \sim \int d^{10}x\sqrt{G^I}\left(\frac{1}{g_I^2}R + \frac{1}{4}\frac{1}{g_I}\left(R_{LMNP}R^{LMNP} - F^a_{MN}F^{a\,MN}\right)\right). \tag{3.6}$$

Now we T-dualize this on a two-torus of volume $V_I$ to type I' theory in eight dimensions. Since the eight-dimensional Newton constant is invariant, we have (modulo factors of 2 and $2\pi$)

$$\frac{V_I}{g_I^2} = \frac{V_{I'}}{g_{I'}^2} \sim \frac{1}{V_I g_I^2} \,, \tag{3.7}$$

and therefore

$$g_I \sim \frac{g_{I'}}{V_{I'}} \tag{3.8}$$

and

$$\frac{V_I}{g_I} \sim \frac{1}{g_{I'}} \,. \tag{3.9}$$

Thus the T-dualized eight-dimensional action is

$$S_{I'} \sim \frac{V_{I'}}{g_{I'}^2}\int d^8x\sqrt{G^{I'}}R + \frac{1}{4}\frac{1}{g_{I'}}\int d^8x\sqrt{G^{I'}}\left(R_{LMNP}R^{LMNP} - F^a_{MN}F^{a\,MN}\right). \tag{3.10}$$

Since T-duality takes D9-branes to D7-branes and O9-planes to O7-planes, one sees that in type I' the $R^2$- and $F^2$-terms come from discs attached to the D7-branes and crosscaps corresopnding to O7-planes. This explains why there is no transverse volume factor $V_{I'}$ in these terms.



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Now, to extract the $N$-dependence of these terms we scale the metric to unit radius, not forgetting to scale $V_{I'}$ as well. Thus

$$\begin{aligned} V_{I'} &\to L^2 V_{I'} \\ d^8x\sqrt{G^{I'}} &\to L^8 d^8x\sqrt{G^{I'}} \\ R &\to L^{-2}R \\ R_{LMNP}R^{LMNP} &\to L^{-4}R_{LMNP}R^{LMNP}\,, \end{aligned} \tag{3.11}$$

and the action becomes

$$S_{I'} \sim \frac{L^8 V_{I'}}{g_{I'}^2}\int d^8x\sqrt{G^{I'}}R + \frac{1}{4}\frac{L^4}{g_{I'}}\int d^8x\sqrt{G^{I'}}\left(R_{LMNP}R^{LMNP} - F^a_{MN}F^{a\,MN}\right). \tag{3.12}$$

Using (2.1) we see that, as anticipated, the string coupling constant drops out and the Einstein term is of order $N^2$ while the curvature squared terms in the effective 7-brane action are of order $N$. For a recent discussion of curvature squared terms in type II D-brane actions see [17].

The precise numerical factors of the five-dimensional action can now also be determined. For the Einstein term, plus the cosmological constant, we have the inverse ten-dimensional Newton constant times the volume $\mathrm{Vol}(X^5) = \mathrm{Vol}(S^5)/2$ times, as we have seen, $L^8$, giving

$$\begin{aligned} S_E &= \frac{1}{16\pi(8\pi^6 g_{I'}^2)}\frac{\pi^3}{2}(8\pi g_{I'}N)^2\int_{AdS_5} d^5x\sqrt{G}(R-2\Lambda) \\ &= \frac{N^2}{4\pi^2}\int_{AdS_5} d^5x\sqrt{G}(R-2\Lambda)\,, \end{aligned} \tag{3.13}$$

where $R$ now denotes the five-dimensional Ricci scalar.

For the Riemann tensor squared term in five dimensions, and related terms arising from the dimensional reduction of the internal and mixed components of this term, the numerical coefficient arises as follows. There is a factor of 1/4 in the ten-dimensional action. It was related by supersymmetry to the anomaly cancelling Green-Schwarz term for the gauge group SO(32). By turning on appropriate Wilson lines in the type I theory, this gauge group can be reduced to $\mathrm{SO}(8)^4$. Upon T-duality, these Wilson lines translate into the positions of the D7-branes in the type I' theory.

Thus there are clusters of 8 D7-branes, each of the clusters located at one of the 4 O7 orientifold planes. As we have seen above, the total $R^2$-term comes from discs attached to D7-branes and crosscaps for the O7-planes, each of the four clusters giving 1/4 of the total contribution. As in the near horizon limit three of these clusters are infinitely far away, only one quarter of this term will be relevant.

Moreover, because of the presence of the orientifold, the volume of the two-torus should be taken to be $(2\pi^2)$ rather than the usual $(2\pi)^2$. Wrapping the D7 branes on



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the $S^3$, the fixed locus of the $\mathbb{Z}_2$ action, produces another contribution $\mathrm{Vol}(S^3)$. This $S^3$ has [11, 9] the standard volume $2\pi^2$. Finally, there is, as we have seen above, a factor of $L^4$ from the scaling of the metric cancelling the $1/g_{I'}$.

Putting everything together, we find that the coefficient of the $R^2$-term (as well as that of the other components of this term and other related 4-derivative terms in the action) is

$$\begin{aligned} S_{R^2} &= \frac{1}{16\pi 8\pi^6}(8\pi N)2\pi^2 2\pi^2 \frac{1}{16}\int_{AdS_5} d^5x\sqrt{G}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots \\ &= \frac{6N}{24\times 16\pi^2}\int_{AdS_5} d^5x\sqrt{G}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots \end{aligned} \tag{3.14}$$

Note the striking similarity of this coefficient with the subleading contribution (2.14) to the trace anomaly. Even though we haven't even begun to calculate the contribution of this term to the trace anomaly, this certainly suggests that we are on the right track.

The same argument shows that the $\mathrm{Tr}\,F^2$-term for the $\mathrm{SO}(8)\subset\mathrm{SO}(8)^4\subset\mathrm{SO}(32)$ gauge fields in the heterotic action, reinstating the factor of 4 we divided by before, gives rise to an order $N$ contribution

$$S_{F^2} = -\frac{N}{16\pi^2}\int_{AdS_5} d^5x\sqrt{G}F^a_{\mu\nu}F^{a\,\mu\nu} + \cdots \tag{3.15}$$

to the bulk action. Alternatively [9], up to an overall normalization, the coefficient of this term could have been deduced from the SO(8) Dirac-Born-Infeld action of the D7-branes. From this point of view it is of course obvious that this is an open string disc contribution and hence of order $N$. The relative factor of 4 between the gravitational and gauge field couplings mirrors that found in [9] for the five-dimensional Chern-Simons terms arising from the D7/O7 RR Chern-Simons couplings.

Note again the striking similarity of this term with the contribution (2.18) of external SO(8) gauge fields to the trace anomaly. Once we have developed the appropriate machinery below, it will be straightforward to verify that (3.15) reproduces exactly the anomaly (2.18).

## 4. Review of the $\mathcal{O}(N^2)$ calculation

### 4.1 The strategy

Before embarking on the calculation of the $\mathcal{O}(N)$ contribution to the trace anomaly, let us quickly review the calculation of the leading $\mathcal{O}(N^2)$ contribution [6].

Because the AdS metric has a second order pole at infinity, AdS space only induces a conformal equivalence class $[g^{(0)}_{ij}]$ of metrics on the boundary. To check for conformal invariance, one chooses a representative $g^{(0)}_{ij}$ acting as a source term for



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the energy-momentum tensor of the boundary theory. The AdS/CFT correspondence predicts that the CFT effective action in the large $N$ supergravity limit is

$$W_{CFT}(g_0) = S_{grav}(g; g_0)\,, \tag{4.1}$$

where $S_{grav}(g; g_0)$ denotes the gravitational action evaluated on a classical configuration which approaches (in the conformal sense) the metric $g^{(0)}_{ij}$ on the boundary. The action is the sum of two terms, the standard bulk action $S_E \sim \int(R - 2\Lambda)$, and a boundary term, involving the trace of the extrinsic curvature of the boundary, required to ensure the absence of boundary terms in the variational principle. To solve the classical equation of motion

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = 0\,, \tag{4.2}$$

with this boundary condition, one can [18, 6] make the following ansatz for the metric,

$$G_{\mu\nu}dx^\mu dx^\nu = \frac{L^2}{4}\frac{d\rho^2}{\rho^2} + \frac{1}{\rho}g_{ij}dx^i dx^j\,, \tag{4.3}$$

with the boundary sitting at $\rho = 0$. We will set $L = 1$ in the following as our scaling arguments use the unit radius metric. The metric $g_{ij}$ has an expansion as [18, 6]

$$g_{ij} = g^{(0)}_{ij} + \rho g^{(2)}_{ij} + \rho^2 g^{(4)}_{ij} + \rho^2 \log\rho h^{(4)}_{ij} + \cdots\,, \tag{4.4}$$

with $g^{(0)}_{ij}$, as above, the chosen boundary value.

Now, for a solution to the classical equations of motion, both the bulk and the boundary term are divergent (the former because for an Einstein manifold the classical Einstein-Hilbert action reduces to a volume integral, and the latter because the induced metric on the boundary is singular). Therefore, one needs to regularize this expression (which, in view of its conjectured relation to the CFT effective action is not surprising). This can be done by introducing a cutoff $\epsilon$ restricting the range of $\rho$ to $\rho \geq \epsilon$. Note that, in agreement with general arguments on holography [19], this bulk IR cutoff corresponds to an UV cutoff in the CFT. Then the regularized CFT effective action $W^\epsilon_{CFT}(g^{(0)})$ is invariant under $\delta g^{(0)} = \lambda g^{(0)}, \delta\epsilon = \lambda\epsilon$.

$W^\epsilon_{CFT}(g^{(0)})$ can be written as a sum of terms diverging as $\epsilon \to 0$, $W^\infty_{CFT}(g^{(0)})$, and a finite term $W^{fin}_{CFT}(g^{(0)})$. The former is a sum of terms which are integrals of local covariant expressions in the boundary metric $g^{(0)}$ and hence they can be removed by local counterterms. Among these terms there is, for $AdS_{d+1}$ with $d$ even, a logarithmically divergent term (which, interestingly enough, does not arise from the logarithmic term in the expansion (4.4) of the metric). In the standard way, removal of this term will then induce a conformal anomaly in the finite part $W^{fin}(g^{(0)})$. The boundary term never contributes to the conformal anomaly (this is a consequence of the fact that the logarithmic term $h^{(4)}_{ij}$ in (4.4) is known to be traceless with respect to $g^{(0)}$ [18]) and we will not consider it in the following.



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### 4.2 Calculation of the $\mathcal{O}(N^2)$ contribution

In the case at hand, the precise form of the anomaly is determined as follows. For an Einstein space with $R_{\mu\nu} = -4g_{\mu\nu}$, the value of the classical lagrangean is $L_c = -8$. The volume element is

$$\sqrt{\det G} = \frac{1}{2}\rho^{-3}\sqrt{\det g}\,, \tag{4.5}$$

where the latter can be expanded as

$$\sqrt{\det g} = \sqrt{\det g^{(0)}}\left(1 + \frac{1}{2}\rho\,\mathrm{Tr}\,g^{(2)} + \frac{1}{8}\rho^2\left[(\mathrm{Tr}\,g^{(2)})^2 - \mathrm{Tr}((g^{(2)})^2)\right]\right) + \cdots\,. \tag{4.6}$$

Here, Tr denotes the trace with respect to the metric $g^{(0)}$ and we have made use of the useful identity [6]

$$\mathrm{Tr}\,g^{(4)} = \frac{1}{4}\,\mathrm{Tr}((g^{(2)})^2)\,. \tag{4.7}$$

By iteratively solving the Einstein equations as a power series in $\rho$, one finds [6]

$$g^{(2)}_{ij} = -\frac{1}{2}\left(r^{(0)}_{ij} - \frac{1}{6}g^{(0)}_{ij}r^{(0)}\right). \tag{4.8}$$

Here $r^{(0)}_{ij}$ denotes the Ricci tensor of $g^{(0)}$ etc. Note that we are using the opposite sign conventions of [6]. Our conventions for the curvature tensor,

$$R^{\lambda}{}_{\sigma\mu\nu} = \partial_\mu\Gamma^{\lambda}{}_{\sigma\nu} - \partial_\nu\Gamma^{\lambda}{}_{\sigma\mu} + \Gamma^{\lambda}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} - \Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\sigma}\,, \tag{4.9}$$

and the Ricci tensor,

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\sigma}R_{\sigma\mu\lambda\nu}\,, \tag{4.10}$$

are such that the curvature of the sphere is positive.

We will need the square of the trace and the trace of the square of this term. One has

$$\begin{aligned}
\mathrm{Tr}\,g^{(2)} &= -\frac{1}{6}r^{(0)} \\
(\mathrm{Tr}\,g^{(2)})^2 &= \frac{1}{36}(r^{(0)})^2 \\
\mathrm{Tr}(g^{(2)})^2 &= \frac{1}{4}\left(r^{(0)}_{ij}r^{(0)ij} - \frac{2}{9}(r^{(0)})^2\right).
\end{aligned} \tag{4.11}$$

In particular, therefore, the order $\rho^2$-term in the expansion (4.6) is

$$(\mathrm{Tr}\,g^{(2)})^2 - \mathrm{Tr}(g^{(2)})^2 = -\frac{1}{4}\left[r^{(0)}_{ij}r^{(0)ij} - \frac{1}{3}(r^{(0)})^2\right]. \tag{4.12}$$

As one obtains a $\rho^{-3}$ from $\sqrt{G}$, it is clear that a logarithmically divergent term will arise only from the term of order $\rho^2$ in (4.6). In particular, we see that for any



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gravitational action including only the Einstein term and a cosmological constant, the leading contribution to the conformal anomaly will be proportional to (4.12). Comparing with the discussion in section 2, we see that this impies $a = c$ to order $N^2$ as (4.12) does not contain a Riem$^2$-term.

Let us now apply this to $AdS_5 \times X^5$ and thus to the leading contribution to the trace anomaly of the $\mathcal{N} = 2$ superconformal field theory considered in [9] and above. Using (3.13), and noting that the factor of $1/2$ in (4.5) is cancelled by a conventional factor of 2 in the definition of the conformal anomaly, one finds that the $\mathcal{O}(N^2)$ conformal anomaly, i.e. the coefficient of the $\log \epsilon$-term, is

$$\frac{N^2}{4\pi^2}(-8)\frac{1}{8}\left[(\mathrm{Tr}\, g^{(2)})^2 - \mathrm{Tr}((g^{(2)})^2)\right] = \frac{N^2}{16\pi^2}\left[r^{(0)}_{ij}r^{(0)ij} - \frac{1}{3}(r^{(0)})^2\right]. \qquad (4.13)$$

This is indeed *precisely* the leading contribution (2.13) to the conformal anomaly calculated on the CFT side.

## 5. The $\mathcal{O}(N)$ contribution

### 5.1 The strategy

Now the strategy for including the Riemann tensor sqared term should be clear. We take the Einstein plus Riemann squared action (3.13) plus (3.14) (possibly also with the $F^2$-term (3.15) - we will comment on the inclusion of this term below)

$$S = \frac{N^2}{4\pi^2}\int d^5x\sqrt{G}(R - 2\Lambda) + \frac{6N}{24\times 16\pi^2}\int d^5x\sqrt{G}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots \qquad (5.1)$$

(plus boundary terms), solve the equations of motion with the given boundary metric $g^{(0)}$, and isolate the log-divergent terms in the action evaluated on this classical solution. Note that, because of the presence of the term $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ in (5.1) this calculation will no longer reduce to just a volume calculation.

In principle, of course, solving the classical equations of motion of this higher-derivative gravity action to the required order in $\rho$ is an unpleasant task. In the present case, however, a drastic simplification is brought about by the fact that we are only interested in the contributions of order $N$ to the classical action. For this, it is sufficient to evaluate the term $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ on the classical solution of the previous section to the original Einstein equation (4.2).

Indeed, as the second term in (5.1) is $1/N$ down with respect to the Einstein term, we can make an ansatz for the solution to the full equations in the form

$$G_{\mu\nu} = G^{(0)}_{\mu\nu} + \frac{1}{N}G^{(1)}_{\mu\nu}\,, \qquad (5.2)$$

where $G^{(0)}_{\mu\nu}$ is a solution of (4.2). Plugging this solution into the Einstein term, i.e. the first term of (5.1), one obtains at order $N^2$ the leading contribution to the anomaly



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calculated in the previous section. A term of order $N$ that could potentially arise as the next term in the expansion is actually zero (because we are expanding about a classical solution to the Einstein action and the boundary term is precisely there to cancel any residual boundary terms). The second term in (5.1) will give a contribution of order $N$ when evaluated on $G^{(0)}_{\mu\nu}$, and any other contributions involving $G^{(1)}_{\mu\nu}$ will be of order 1 or lower.

Therefore, to find the order $N$ contributions to the trace anomaly, we need to

1. calculate the Riemann curvature tensor of the metric (4.3), with $g_{ij}$ given by (4.4) and (4.8), as a function of $\rho$, and

2. then determine the order $\rho^{-1}$-terms in the $\rho$-expansion of

$$\sqrt{\det G} G^{\alpha\mu} G^{\beta\nu} G^{\gamma\lambda} G^{\delta\sigma} R_{\alpha\beta\gamma\delta} R_{\mu\nu\lambda\sigma} \,. \tag{5.3}$$

Since we know that $\sqrt{\det G} \sim \rho^{-3}\sqrt{\det g}$ (4.5), this means that we need to pick up the order $\rho^2$-terms from

$$\sqrt{\det g} G^{\alpha\mu} G^{\beta\nu} G^{\gamma\lambda} G^{\delta\sigma} R_{\alpha\beta\gamma\delta} R_{\mu\nu\lambda\sigma} \,, \tag{5.4}$$

Here the $\rho$-expansions of the curvature, of $\sqrt{\det g}$ and of the inverse metric have to be considered.[2]

If one also includes the $F^2$-term (3.15), then in principle one would of corse have to solve the coupled Einstein-Yang-Mills equations. But as the $F^2$-term is also of order $N$ the same argument as above shows that the resulting subleading corrections to the metric are again irrelevant. As regards the equation of motion for $F$ itself, we will see below that only the boundary value of $F$ contributes so we do not have to solve these equations either.

### 5.2 External gauge fields

Let us begin with the external gauge field contribution (2.18) to the anomaly as it is by far the simplest contribution to determine (much simpler, in fact, than even the leading $\mathcal{O}(N^2)$ contribution to the anomaly discussed above).

As the above discussion shows, we need to pick up the order $\rho^2$-terms of

$$\sqrt{\det g} G^{\alpha\mu} G^{\beta\nu} F^a_{\alpha\beta} F^a_{\mu\nu} \,. \tag{5.5}$$

Now the components of the inverse metric are $G^{\rho\rho} = 4\rho^2$, $G^{ij} = \rho g^{ij}$, where $g^{ij}$ has the expansion

$$g^{ij} = g^{(0)ij} - \rho g^{(2)ij} + \rho^2 \left( \left( (g^{(2)})^2 \right)^{ij} - g^{(4)ij} \right) + \cdots \,, \tag{5.6}$$

[2]Similar calculations have recently also been performed in [20], however with the diametrically opposite motivation of trying to reproduce the leading $\mathcal{O}(N^2)$ contribution to the anomaly from a higher derivative action ...



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where indices are raised with $g^{(0)ij}$. As the two inverse metrics contribute at least a factor of $\rho$ each, the only contribution to the anomaly arises from

$$\sqrt{\det g^{(0)}} g^{(0)ik} g^{(0)jl} F^{(0)a}_{ij} F^{(0)a}_{kl} \,, \tag{5.7}$$

where $F^{(0)}_{ij}$ is the boundary value of the gauge field

$$F_{ij} = F^{(0)}_{ij} + \mathcal{O}(\rho) \,. \tag{5.8}$$

In [9] the relation betwen the bulk supergravity and boundary SCFT SO(8)-generators, in the fundamental representation **8**, was determined from the AdS/CFT correspondence. Using this result, one obtains that $T(\mathbf{8})$, appearing in the field theoretic expression (2.18), is equal to 1. Therefore, (3.15) gives precisely the external gauge field contribution (2.18) to the trace anomaly.

Alternatively, this term could have been deduced (in the abelian, non-interacting case) by following the prescription in [3]: On-shell, the bulk Maxwell action reduces to a boundary term, and this boundary term can be evaluated in terms of Witten's bulk-to-boundary Green's functions, extracting the local term (relevant to the anomaly) in the end.

More directly, one can proceed locally, i.e. without using Green's functions, by solving the Maxwell equations in a $\rho$-expansion as was done for the Einstein equations in [6]. From this vantage point, the logarithmic divergence arises directly in the boundary term $\sim \int A_i \partial_\rho A_i$ because a term of order $\rho \log \rho$ in the $\rho$-expansion of $A_i$ turns out to be required to solve the bulk Maxwell equations (cf. the $\rho^2 \log \rho$-term in the expansion (4.4) of the metric, required for the same reason).

### 5.3 The curvature tensor

As three different metrics appear here, $G_{\mu\nu}$, $g_{ij}$ and $g^{(0)}_{ij}$, we will correspondingly denote their curvature tensors by $R^{\mu}{}_{\nu\lambda\rho}$, $r^{i}{}_{jkl}$, $r^{(0)i}{}_{jkl}$. $\nabla$ will denote the covariant derivative compatible with $g_{ij}$, $\nabla^{(0)}$ that compatible with $g^{(0)}_{ij}$. $\rho$-derivatives will be denoted by a prime.

The ubiquitous combination $g_{ij} - \rho g'_{ij}$, which we will abbreviate to $k_{ij}$ in the following, contains no terms linear in $\rho$. Up to $\rho^2 \log \rho$-terms one has

$$k_{ij} \equiv g_{ij} - \rho g'_{ij} = g^{(0)}_{ij} - \rho^2 (g^{(4)} + h^{(4)}) + \cdots \tag{5.9}$$

We will sometimes also abbreviate $g^{(4)} + h^{(4)} = f^{(4)}$.

For the curvature tensor one then finds

$$\begin{aligned} R_{ijkl} &= \rho^{-1}[r_{ijkl} + \rho^{-1}(k_{il}k_{jk} - k_{ik}k_{jl})] \\ R_{\rho ijk} &= \frac{1}{2}\rho^{-2}(\nabla_j k_{ik} - \nabla_k k_{ij}) \\ R^{\rho}{}_{i\rho j} &= -2\rho^{-1}(k_{ij} - \rho k'_{ij}) + \rho^{-1} k^2_{ij} \,. \end{aligned} \tag{5.10}$$

where in the last line the product is taken with respect to the metric $g_{ij}$.

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We will need the $\rho$-expansion of these curvature tensors. For $R_{ijkl}$, we need to expand $r_{ijkl}$ as well as the other terms. Symbolically we have

$$r^{i}_{\ jkl} = r^{(0)i}_{\ \ jkl} + \rho\left(\nabla^{(0)}_{k}\delta\Gamma^{i}_{\ jl} - \nabla^{(0)}_{l}\delta\Gamma^{i}_{\ jk}\right) + \cdots\,, \tag{5.11}$$

where we do not need to know the precise form of the $\delta\Gamma$'s. Using this and the definition of $k_{ij}$ one finds

$$\begin{aligned} R_{ijkl} &= G_{in}R^{n}_{\ jkl} = \rho^{-1}g_{in}R^{n}_{\ jkl} \\ R_{ijkl} &= \rho^{-2}\left[g^{(0)}_{il}g^{(0)}_{jk} - g^{(0)}_{ik}g^{(0)}_{jl}\right] + \rho^{-1}r^{(0)}_{ijkl} + \\ &\quad +\rho^{0}\left[g^{(0)}_{ik}f^{(4)}_{jl} + g^{(0)}_{jl}f^{(4)}_{ik} - g^{(0)}_{il}f^{(4)}_{jk} - g^{(0)}_{jk}f^{(4)}_{il}\right] + \\ &\quad +\rho^{0}\left[\nabla^{0}_{k}\delta\Gamma_{ijl} - \nabla^{(0)}_{l}\delta\Gamma_{ijk}\right] + \rho^{0}\left[g^{(2)}_{in}r^{(0)n}_{\ \ jkl}\right] + \mathcal{O}(\rho) \end{aligned} \tag{5.12}$$

$R_{\rho ijk}$ is simpler, we just keep the first term (and not even that one will contribute as we will see),

$$R_{\rho ijk} = \rho^{-1}\left[-\frac{1}{2}\left(\nabla_{j}g^{(2)}_{ik} - \nabla_{k}g^{(2)}_{ij}\right)\right] + \mathcal{O}(1) \tag{5.13}$$

For $R^{\rho}_{\ i\rho j}$, one has

$$R^{\rho}_{\ i\rho j} = \rho^{-1}\left[-g^{(0)}_{ij}\right] + \rho^{0}\left[-g^{(2)}_{ij}\right] + \rho^{+1}\left[-5f^{(4)}_{ij} + (g^{(2)})^{2}_{ij}\right] + \mathcal{O}(\rho^{2})\,, \tag{5.14}$$

where now, of course, in the last line the product is taken with respect to $g^{(0)}$.

As mentioned above, we need to pick up the order $\rho^2$-terms from

$$\sqrt{\det g}G^{\alpha\mu}G^{\beta\nu}G^{\gamma\lambda}G^{\delta\sigma}R_{\alpha\beta\gamma\delta}R_{\mu\nu\lambda\sigma}\,, \tag{5.15}$$

Let us deal with $R_{\rho ijk}$ first. In that case, the factor entering the contractions is

$$\sqrt{\det g}G^{\rho\rho}G^{im}G^{jn}G^{kp}\,. \tag{5.16}$$

This will contribute at least $\rho^2 \times \rho^3 = \rho^5$, but the highest negative power of $\rho$ that can arise from the square of $R_{\rho ijk}$ is $\rho^{-2}$, giving an overall $\rho^3$ and therefore no contribution to the anomaly.

For $R_{ijkl}$ we have

$$\sqrt{\det g}G^{im}G^{jn}G^{kp}G^{lq}\,. \tag{5.17}$$

This contributes $\rho^4$ and higher powers. But the highest negative power arising from $R^2_{ijkl}$ is $\rho^{-4}$. Hence here terms of order $\rho^4$, $\rho^5$ and $\rho^6$ in the expansion of the con-



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traction/volume factor (5.17) are relevant. At order $\rho^n$, $n = 4, 5, 6$ respectively, one has:

$$\begin{aligned}
\rho^4 :\ & \sqrt{g^{(0)}} g^{(0)im} g^{(0)jn} g^{(0)kp} g^{(0)lq} \\
\rho^5 :\ & \sqrt{g^{(0)}} \frac{1}{2} \operatorname{Tr} g^{(2)} g^{(0)im} g^{(0)jn} g^{(0)kp} g^{(0)lq} - 4\sqrt{g^{(0)}} g^{(2)im} g^{(0)jn} g^{(0)kp} g^{(0)lq} \\
\rho^6 :\ & \sqrt{g^{(0)}} \frac{1}{8} \left[ (\operatorname{Tr} g^{(2)})^2 - \operatorname{Tr}(g^{(2)})^2 \right] g^{(0)im} g^{(0)jn} g^{(0)kp} g^{(0)lq} - \\
& - 4\sqrt{g^{(0)}} \frac{1}{2} \operatorname{Tr} g^{(2)} g^{(2)im} g^{(0)jn} g^{(0)kp} g^{(0)lq} + \\
& + 4\sqrt{g^{(0)}} \left[ ((g^{(2)})^2 - g^{(4)})^{im} \right] g^{(0)jn} g^{(0)kp} g^{(0)lq} + \\
& + 2\sqrt{g^{(0)}} g^{(2)im} g^{(2)jn} g^{(0)kp} g^{(0)lq} 4\sqrt{g^{(0)}} g^{(2)im} g^{(0)jn} g^{(2)kp} g^{(0)lq} \,. \qquad (5.18)
\end{aligned}$$

Finally, for $R^{\rho}_{i\rho j}$, the structure is

$$\sqrt{\det g} G^{im} G^{jn} \,. \qquad (5.19)$$

This will be of order $\rho^2$ and higher. On the other hand, the square of the curvature tensor gives terms of order $\rho^{-2}$ and higher. Hence in the expansion of the contraction/volume factor (5.19), terms of order $\rho^2$, $\rho^3$, $\rho^4$ are relevant. These are

$$\begin{aligned}
\rho^2 :\ & \sqrt{g^{(0)}} g^{(0)im} g^{(0)jn} \\
\rho^3 :\ & \sqrt{g^{(0)}} \frac{1}{2} \operatorname{Tr} g^{(2)} g^{(0)im} g^{(0)jn} - 2\sqrt{g^{(0)}} g^{(2)im} g^{(0)jn} \\
\rho^4 :\ & \sqrt{g^{(0)}} \frac{1}{8} \left[ (\operatorname{Tr} g^{(2)})^2 - \operatorname{Tr}(g^{(2)})^2 \right] g^{(0)im} g^{(0)jn} - 2\sqrt{g^{(0)}} \frac{1}{2} \operatorname{Tr} g^{(2)} g^{(2)im} g^{(0)jn} + \\
& + 2\sqrt{g^{(0)}} \left[ ((g^{(2)})^2 - g^{(4)})^{im} \right] g^{(0)jn} + \sqrt{g^{(0)}} g^{(2)im} g^{(2)jn} \,. \qquad (5.20)
\end{aligned}$$

### 5.4 Contributions from $R_{ijkl}$

Let us call the five contributions in (5.12) $I$, $II$, $III$, $IV$ and $V$. Three terms contribute to the $\rho^4$-term of (5.18), namely $II \times II$, $I \times III$ and $I \times V$. $I \times IV$ only contributes a total derivative of a covariant quantity and can therefore be cancelled by the variation of a local counterterm. Using the tracelessness of $h^{(4)}$ one sees that $h^{(4)}$ will not contribute either. The other terms give

$$\begin{aligned}
\rho^4 \,,\ II \times II \qquad & r^{(0)}_{ijkl} r^{(0)ijkl} \,, \\
\rho^4 \,,\ I \times III \qquad & -6 \operatorname{Tr}(g^{(2)})^2 \,, \\
\rho^4 \,,\ I \times V \qquad & -4 g^{(2)}_{ij} r^{(0)ij} \,, \qquad (5.21)
\end{aligned}$$

where we have used (4.7). The two terms of order $\rho^5$ in (5.18) need to be paired with $I \times II$:

$$\rho^5 \,,\ I \times II \qquad -2r^{(0)} \operatorname{Tr} g^{(2)} + 16 g^{(2)ij} r^{(0)}_{ij} \,. \qquad (5.22)$$



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The terms of order $\rho^6$ in (5.18) need to be paired with $I \times I$. From the first three terms of order $\rho^6$ we get

$$\rho^6\,,\ I\times I \qquad -9(\mathrm{Tr}\, g^{(2)})^2 + 15\,\mathrm{Tr}(g^{(2)})^2\,. \tag{5.23}$$

The fourth and fifth term give

$$\begin{aligned}\rho^6\,,\ I\times I \qquad & 4(\mathrm{Tr}\, g^{(2)})^2 - 4\,\mathrm{Tr}(g^{(2)})^2\,,\\ & 4(\mathrm{Tr}\, g^{(2)})^2 + 8\,\mathrm{Tr}(g^{(2)})^2\,.\end{aligned} \tag{5.24}$$

Adding all this up, we find the subtotal from $R_{ijkl}$ to be

$$\begin{aligned}r^{(0)}_{ijkl}r^{(0)ijkl} - 2r^{(0)}\,\mathrm{Tr}\, g^{(2)} + 12 g^{(2)ij}r^{(0)}_{ij} - (\mathrm{Tr}\, g^{(2)})^2 + 13\,\mathrm{Tr}(g^{(2)})^2 = \\ = r^{(0)}_{ijkl}r^{(0)ijkl} - \frac{11}{4}r^{(0)}_{ij}r^{(0)ij} + \frac{7}{12}(r^{(0)})^2\,.\end{aligned} \tag{5.25}$$

### 5.5 Contributions from $R^{\rho}_{i\rho j}$

We proceed as above. The three terms of (5.20) we call $I$, $II$, $III$. Every contribution has to be multiplied by four, because there are four components of the Riemann tensor with two $\rho$'s.

$$\begin{aligned}\rho^2\,,\ II\times II \qquad & 4\,\mathrm{Tr}(g^{(2)})^2\,,\\ \rho^2\,,\ I\times III2 \qquad & \mathrm{Tr}(g^{(2)})^2\,,\\ \rho^3\,,\ I\times II \qquad & 4(\mathrm{Tr}\, g^{(2)})^2 - 16\,\mathrm{Tr}(g^{(2)})^2\,.\end{aligned} \tag{5.26}$$

There are four terms of order $\rho^4$ in (5.20), to be paired with $I \times I$. These give

$$\begin{aligned}\rho^4\,,\ I\times I \qquad & 2(\mathrm{Tr}\, g^{(2)})^2 - 2\,\mathrm{Tr}(g^{(2)})^2\,,\\ & 4\,\mathrm{Tr}(g^{(2)})^2\,,\\ & 6\,\mathrm{Tr}(g^{(2)})^2\,,\\ & -4(\mathrm{Tr}\, g^{(2)})^2\,.\end{aligned} \tag{5.27}$$

Adding all these up, one gets

$$2(\mathrm{Tr}\, g^{(2)})^2 - 2\,\mathrm{Tr}(g^{(2)})^2 = -\frac{1}{2}r^{(0)}_{ij}r^{(0)ij} + \frac{1}{6}(r^{(0)})^2\,. \tag{5.28}$$

### 5.6 The total $\mathcal{O}(N)$ contribution to the trace anomaly

Adding up all the above contributions, and remebering the prefactor in (3.14), we find that supergravity predicts the $\mathcal{O}(N)$ contribution to the trace anomaly to be

$$\frac{6N}{24\times 16\pi^2}\left[r^{(0)}_{ijkl}r^{(0)ijkl} - \frac{13}{4}r^{(0)}_{ij}r^{(0)ij} + \frac{3}{4}(r^{(0)})^2\right]. \tag{5.29}$$



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A glance at (2.14) shows that this does not yet look particularly encouraging. However, let us split these terms as

$$\frac{6N}{24\times 16\pi^2}[r^{(0)}_{ijkl}r^{(0)ijkl}+2r^{(0)}_{ij}r^{(0)ij}-(r^{(0)})^2] \\ -\frac{6N}{24\times 16\pi^2}\frac{21}{4}\left[r^{(0)}_{ij}r^{(0)ij}-\frac{1}{3}(r^{(0)})^2\right]. \tag{5.30}$$

We see that the first term reproduces *precisely* the subleading contribution (2.14) to the conformal anomaly, in particular with the crucial term proportional to $(a-c)$. The second (error) term, on the other hand, is exactly (and this is an important check on our calculation) of the form of a volume contribution (4.12), just like the leading $\mathcal{O}(N^2)$-term. We will say more about the possible origin of this volume term below.

## 6. $\mathcal{O}(N)$ corrections for SO and Sp $\mathcal{N}=4$ theories

In this section we will briefly discuss another class of models which, at first sight, seems to present a puzzle. Looking back at the $\mathcal{N}=4$, SU($N$) trace anomaly (2.7), we see that there is a tree-level contribution, determined in [6], no term of order $N$ but an $\mathcal{O}(1)$ correction that ought to arise from a string one-loop calculation. For other gauge groups $G$, however, the situation is different. In general, one has

$$\langle T^{\mu}_{\mu}\rangle=\frac{\dim(G)}{32\pi^2}\left[\mathrm{Ric}^2-\frac{1}{3}\mathrm{R}^2\right]. \tag{6.1}$$

In particular, for orthogonal (symplectic) gauge groups SO($N$) (Sp($N/2$) for $N$ even), $\dim(G)$ contains both quadratic and linear terms in $N$,

$$\dim(\mathrm{SO}(2k))=k(2k-1) \\ \dim(\mathrm{SO}(2k+1))=\dim(\mathrm{Sp}(k))=k(2k+1), \tag{6.2}$$

and we want to understand the origin of these linear terms in the AdS/CFT correspondence.

$\mathcal{N}=4$ theories with gauge groups SO($N$), Sp($N/2$) can be realized as the low-energy dynamics of $N$ parallel D3-branes at an orientifold O3-plane [21], i.e. with the branes sitting at the singularity of a transverse $\mathbb{R}^6/\mathbb{Z}_2$. Here $\mathbb{Z}_2$ acts as $\vec{x}\to-\vec{x}$ for $\vec{x}\in\mathbb{R}^6$. Note that, because of the non-compactness of the transverse space, the number of D3-branes is not fixed by RR tadpole cancellation.

This strongly suggests [22] that a string theory dual to these theories is given by type IIB string theory on an $AdS_5\times\mathbb{RP}^5$ orientifold.

Clearly, unlike for the $\mathcal{N}=2$ theory we discussed above, now there are no branes wrapping the entire $AdS_5$. So where are the terms linear in $N$ going to come from?



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The answer is: from the classical Einstein action itself. The reason for this is that O$p$-planes themeselves are carriers of RR-charge, and hence the numerical value of $N$ appearing in the classical D$p$-brane solutions (in the coefficient of the term $r^{p-7}$ in the corresponding harmonic function, $r$ being the transverse distance from the brane), will be shifted in the presence of an orientifold O$p$-plane. In particular, O3-planes carry fractional RR charge $\pm 1/4$ [21, 23]. With the minus sign, one obtains SO($N$), and with the plus sign, for $N$ even, Sp($N/2$) gauge theories. In fact, at least prior to taking the near-horizon limit, in the coefficient of the leading order $\mathcal{O}((\sqrt{\alpha'}/r)^{7-p})$ correction to the classical D$p$-brane solution, this $N$-independent term arises from the addition of the crosscap $\mathbb{RP}^2$ orientifold contribution (of order $g_s$) to the disc D-brane diagram (of order $g_s N$).

This should extend to all orders to reproduce the expected result so that, in the AdS-limit, the net-effect of the presence of the O3-plane is to replace $N$ as appearing in (2.1) by

$$N \to \frac{N}{2} \pm \frac{1}{4}\,, \tag{6.3}$$

where we took also into account that only $N/2$ of the $N$ D3-branes lie on $\mathbb{R}^6/\mathbb{Z}_2$. Consequently

$$L^4 = 8\pi g_s \left(\frac{N}{2} \pm \frac{1}{4}\right)\,. \tag{6.4}$$

Repeating the calculation in (3.13) and section 4 for the leading contribution to the trace anomaly, we now find

$$\begin{aligned}\langle T^\mu_\mu \rangle &= \frac{1}{16\pi 8\pi^6 g_s^2}\mathrm{Vol}(\mathbb{RP}^5) L^8 \int_{AdS_5} d^5x\sqrt{G}(R-2\Lambda) \\ &= \frac{1}{16\pi 8\pi^6}\frac{\pi^3}{2} 64\pi^2 \left(\frac{N}{2}\pm\frac{1}{4}\right)^2 \frac{1}{4}\left[\mathrm{Ric}^2 - \frac{1}{3}\mathrm{R}^2\right] \\ &= \frac{1}{32\pi^2}\frac{N(N\pm 1)+\frac{1}{8}}{2}\left[\mathrm{Ric}^2 - \frac{1}{3}\mathrm{R}^2\right]\,.\end{aligned} \tag{6.5}$$

Comparing with (6.2), we see that to order $N$ this agrees exactly with the trace anomaly formula (6.1) for $G = \mathrm{SO}(N)$ and $G = \mathrm{Sp}(N/2)$. In these cases we have therefore been able to reproduce both the leading and the subleading order $N$ corrections directly from the classical Einstein action by taking into account the fractional RR charge of the O3-plane.

## 7. Discussion

We have shown that supergravity calculations with higher-derivative actions are capable of reproducing the subleading corrections to the CFT trace anomaly. In this particular example, on the basis of the results of [9] this was to be expected on general grounds since supersymmetry relates the chiral and trace anomalies. Nevertheless,



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we find it quite remarkable that the somewhat messy (even though straightforward) classical calculations performed above conspire to give precisely the correct result for the trace anomaly in the end.

As regards the 'missing' volume contribution, we have of course attempted to determine this in a variety of ways but it seems to us that a definitive answer requires a better understanding of $\alpha'$-corrections and supersymmetrization of the type I' effective action.

As the expansion (4.6) produces 1/8 times (4.12), we see that what we are missing is an effective cosmological constant term

$$-\frac{168 \times 6N}{24 \times 16\pi^2} \int_{AdS_5} \sqrt{G} d^5x \,. \tag{7.1}$$

There are many possible terms that can contribute to this cosmological constant. For instance, we have so far neglected the contributions of the internal and mixed components of $R_{LMNP}R^{LMNP}$. These contributions can in principle be determined either from duality arguments or by a direct two-point function calculation on the type I' side. There may also be four-derivative terms of the metric involving the squares of the Ricci tensor or Ricci scalar (perhaps in the form of the familiar Gauss-Bonnet combination). Such terms are afflicted by the usual field redefinition ambiguities. Finally, there may also be terms involving $(F^{(5)})^4$ and mixed terms of the type $R(F^{(5)})^2$. The former correspond to four-point functions on the type I' side and a direct calculation of these terms, although possible in principle, is somewhat cumbersome.

One might have hoped to be able to invoke heterotic - type I - type I' duality once more to fix these terms. For instance, in the heterotic string it is known [13] that supersymmetry forces the CP-even four-derivative terms involving $g_{MN}$ and $B_{MN}$ to appear as curvature-squared terms of the connection with torsion $H = dB + \cdots$. As $H$ eventually dualizes to $F^{(5)}$ in the type I' theory, this is the sort of restrictive structure one might have hoped for. However, chasing these terms through the dualities is somewhat problematic.

For one, as one is taking a large volume limit on the type I' side, this corresponds to a small two-torus on the type I side, and thus it appears that winding mode contributions in Type I need also be considered. Also, at a purely classical level, in order to T-dualize the D7/O7/D3 configuration underlying the $\mathcal{N} = 2$ theory we have been considering to a type I confiiguration of D9/D5 branes one needs to delocalize it in the transverse directions. But if one does that, it will no longer have the same near-horizon limit (T-duality and near-horizon limits do not commute). Conversely, we have been unable to find a classical type I solution which gives the desired configuration on the type I' side and which could have been used to calculate the effective cosmological constant directly on the type I side.



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All this just confirms the general picture that appears to be emerging from the work done on the AdS/CFT correspondence, namely that whatever can be checked reliably confirms the conjectured correspondence, but that even simple (one-loop, anomaly) field theory calculations are difficult to reproduce on the AdS side. Clearly, what is required among other things is a better understanding of string theory with RR backgrounds.

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